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FREE OSCILLATIONS OF LAYERED ELASTIC COMPOSITE SHELLS OF REVOLUTION

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An effective algorithm for the numerical solution of linear boundary-value problems of the stability of multilayered composite shells of revolution combining the Bubnov–Galerkin method for systems of Fredholm integral equations of the second kind with the generalized form of the invariant imbedding method has been developed in [1].

In this paper, the algorithm is extended to the class of problems on the free oscillations of shells of revolution and is used in the numerical determination of the natural frequencies and oscillation forms of a layered composite circular conic and truncated shell which is firmly fixed. A comparative analysis of the calculation results obtained on the basis of classical and nonclassical [2] differential equations for the dynamics of layered shells is carried out. This enabled identification and estimation of the influence of transverse shear deformations on the natural frequencies and forms of oscillations. The results obtained allow a conclusion on the effectiveness of the algorithm in problems of shell dynamics.

1. Numerical Determination of natural Frequencies and Oscillation Forms of Layered Shells of Revolution. Let us consider a problem on free steady-state harmonic oscillations of a thin-walled elastic layered composite shell of revolution, where the reinforcement structure of the layers is independent of the angular coordinate. The differential equations and boundary conditions for this problem are obtained (see [3–7]) from the corresponding linearized equations

$$(D + \Lambda\partial^2/\partial t^2)\,\mathbf{U} = \mathbf{F}$$

and from the boundary conditions

$$l_\Gamma \mathbf{U} = \mathbf{P}$$

for the dynamics of layered shells. Taking the components of the external surface and contour loads equal to zero and after the transformation (ω is the frequency parameter)

$$\frac{\partial^2}{\partial t^2} \to -\omega^2 \tag{1.1}$$

we get a boundary-value problem on the eigenvalues

$$D\mathbf{U} = \omega^2 \Lambda \mathbf{U}, \quad l_\Gamma \mathbf{U} = 0. \tag{1.2}$$

The determination of the spectrum of free oscillations of the thin-walled layered system is reduced to the integration of this problem. Here $\mathbf{U}$ is the vector of characteristics of the strain-stressed state of the shell; D, Λ, l_Γ are linear differential operators with two independent variables: the angular variable φ ($-\pi \leqslant \varphi \leqslant \pi$) and the meridional variable x ($0 \leqslant x \leqslant 1$). The explicit form of these operators is given below for a particular case. Separation of the angular coordinate φ in (1.2), which is carried out (see [3–7]) by expansion of the solution into a Fourier series by using the system of trigonometric functions 1, cos $n\varphi$, sin $n\varphi$ ($1 \leqslant n < \infty$) leads to a linear boundary-value problem on eigenvalues for the system of ordinary differential equations

$$\mathbf{y}'(x) = A_n(x)\,\mathbf{y}(x) + \omega^2 B_n(x)\,\mathbf{y}(x), \quad M\mathbf{y}(0) = 0, \quad N\mathbf{y}(1) = 0. \tag{1.3}$$

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Here $\mathbf{y}(x)$ is the $2s$-dimensional vector of the kinematic and force characteristics of the strain-stressed state of the shell; $A_n(x)$, $B_n(x)$ are the $2s \times 2s$ matrices continuous in the segment [0,1], containing the integer parameter n of the peripheral wave formation; and M, N are $s \times 2s$ matrices.

The boundary-value problem

$$\mathbf{z}'(x) = A_n(x)\,\mathbf{z}(x) + \mathbf{f}(x), \quad M\,\mathbf{z}(0) = 0, \quad N\,\mathbf{z}(1) = 0 \tag{1.4}$$

is considered together with (1.3) for the $2s$-dimensional vector $\mathbf{z}(x)$. Let $G_n(x,p)$ be the Green matrix [8] of this boundary-value problem. It is known [8] that the solution to problem (1.4) for each vector $\mathbf{f}(x)$ continuous in the segment [0,1] can be written in the form

$$\mathbf{z}(x) = \int_0^1 G_n(x,p)\,\mathbf{f}(p)\,dp\,, \tag{1.5}$$

and the problem on the eigenvalues of (1.3) is equivalent to the problem of determination of characteristic numbers and eigenvectors of the linear homogeneous system of Fredholm integral equations of the second kind

$$\mathbf{y}(x) - \omega^2 \int_0^1 G_n(x,p)\,B_n(p)\,\mathbf{y}(p)\,dp = 0\,. \tag{1.6}$$

The solution to problem (1.6) will be constructed in the space $L_2(0,1)$ of quadratically summable $2s$-dimensional vectors by the Bubnov–Galerkin method [9]. In accordance with this method, the complete linearly independent system of elements $\{\boldsymbol{\psi}_k(x)\}_{k=1}^{\infty}$ will be chosen in the space $L_2(0,1)$. The vectors $\boldsymbol{\psi}_k(x)$ are taken to be continuous and orthonormalized. The approximate solution $\mathbf{y}_L(x)$ of the system of integral equations (1.6) is constructed in the form

$$\mathbf{y}_L(x) = \sum_{k=1}^{L} c_k\,\boldsymbol{\psi}_k(x)\,. \tag{1.7}$$

Substituting solution (1.7) into system (1.6) and requiring that the residual be orthogonal to the coordinate vectors $\boldsymbol{\psi}_1(x), \boldsymbol{\psi}_2(x), \ldots, \boldsymbol{\psi}_L(x)$, we obtain an algebraic system of linear homogeneous equations for determining the coefficients $c_1, c_2, \ldots, c_L$:

$$c_i - \omega^2 \sum_{k=1}^{L} c_k \int_0^1 \Big(\int_0^1 G_n(x,p)\,B_n(p)\,\boldsymbol{\psi}_k(p)\,dp \Big)\,\boldsymbol{\psi}_i(x)\,dx = 0 \qquad (i = 1,2,\ldots,L)\,. \tag{1.8}$$

Only nontrivial solutions to system (1.8) are of physical interest. The determination of such solutions — eigenvectors and the corresponding eigenvalues of the matrix of coefficients for this system — involves, in the general case, the use of numerical methods. At present, the methods of numerical solution to the algebraic eigenvalue problem have been developed rather well (see, for example, [10]). If this problem is solved and $\mu_1^{(n)}, \mu_2^{(n)}, \ldots, \mu_L^{(n)}$ are the eigenvalues of the matrix of coefficients from system (1.8), then the equalities

$$\omega_1^{(n)} = \sqrt{1/\mu_1^{(n)}},\ \omega_2^{(n)} = \sqrt{1/\mu_2^{(n)}}, \qquad \ldots, \qquad \omega_L^{(n)} = \sqrt{1/\mu_L^{(n)}} \tag{1.9}$$

serve to determine the approximate values of the lower natural frequencies of the shell corresponding to the given value of the parameter n. Sufficient accuracy in representing these values in the form (1.9) is ensured by the convergence of the Bubnov–Galerkin process for the Fredholm integral equations of the second kind (see [9]) and by the appropriate choice of the value for the parameter L.

The volume of computations can be substantially reduced if we take into account the fact that many columns of the matrix $B_n(x)$ in problems of shell dynamics are zero. Let K be a set of numbers of zero columns and J, a set of numbers of nonzero columns of this matrix, such that $K \cup J = \{1,2,\ldots,2s\}$, $K \cap J = \varnothing$. It is clear from (1.8) that, in constructing the basic system $\{\boldsymbol{\psi}_k(x)\}_{k=1}^{\infty}$, it will suffice to require its completeness in the class $L_J \subset L_2(0,1)$ of $2s$-dimensional vector-functions whose K-coordinates are equal to zero. This basic system is sufficient for the approximation of J-coordinates of eigenvectors from (1.6) and for the correct determination of natural frequencies. If it is also necessary to calculate the K-coordinates of the eigenvectors, then the system $\{\boldsymbol{\psi}_k(x)\}_{k=1}^{\infty}$ should be extended to a system which is complete in the entire space $L_2(0,1)$ and Eqs. (1.7) and (1.8) at $\omega = \omega_1^{(n)}, \omega = \omega_2^{(n)}, \ldots, \omega = \omega_L^{(n)}$ should be used.

For the computation of the matrix of coefficients for the linear algebraic system (1.8), we note that the internal integrals in (1.8) [$2s$-dimensional vectors $\mathbf{z}_1(x)$, $\mathbf{z}_2(x)$, ..., $\mathbf{z}_L(x)$] are, due to (1.4) and (1.5), the solutions to the following boundary-value problems ($k = 1, 2, \ldots, L$):

$$\mathbf{z}_k'(x) = A_n(x)\,\mathbf{z}_k(x) + B_n(x)\,\boldsymbol{\psi}_k(x)\,, \qquad M\,\mathbf{z}_k(0) = 0\,, \qquad N\,\mathbf{z}_k(1) = 0\,. \tag{1.10}$$

Let us combine the $2s$-dimensional vector-columns $\mathbf{z}_1(x), \mathbf{z}_2(x), \ldots, \mathbf{z}_L(x)$ and $B_n(x)\,\boldsymbol{\psi}_1(x)$, $B_n(x)\,\boldsymbol{\psi}_2(x), \ldots,$ $B_n(x)\,\boldsymbol{\psi}_L(x)$ into the $2s \times L$ matrices $Z(x)$ and $F_n(x)$:

$$Z(x) = \|\mathbf{z}_1(x), \mathbf{z}_2(x), \ldots, \mathbf{z}_L(x)\|\,, \quad F_n(x) = B_n(x)\|\boldsymbol{\psi}_1(x), \boldsymbol{\psi}_2(x), \ldots, \boldsymbol{\psi}_L(x)\|\,.$$

Now the family of boundary-value problems (1.10) can be formulated as the following boundary-value problem

$$Z'(x) = A_n(x)Z(x) + F_n(x)\,, \quad MZ(0) = O_{s\times L}\,, \quad NZ(1) = O_{s\times L} \tag{1.11}$$

for the determination of the $2s \times L$ matrix $Z(x)$. If this problem is solved and the matrix $Z(x)$ is found, then the external integrals in system (1.8) are computed by using some quadrature formula of numerical integration. This completes the determination of the matrix of coefficients for this system.

Let us point out two essential peculiarities which should be taken into account in choosing the numerical integration method for the boundary-value problem (1.11): the matrix structure of its solution and the strong instability inherent in the nonclassical differential equations of the theory of layered shells (see [11]). The invariant imbedding method in its generalized form is an effective method of numerical integration for such problems (see [1, 12]). The computational experience accumulated (see [11–14]) makes it possible to recommend this modification of the imbedding method for wide use in problems of strength, stability, and dynamics of shells.

2. Differential Equations for Free Oscillations of a Multilayered Orthotropic Conic Shell. Let us consider an orthotropic circular conic truncated shell of thickness h consisting of m composite layers with a fibrous structure. Let 2α be the cone angle, $s = x^1$ the distance along the generatrix of the cone from its top ($0 < a \leqslant s \leqslant b$), and $\varphi = x^2$ the angular coordinate ($0 \leqslant \varphi \leqslant 2\pi$). The Lamé parameters A_1, A_2 and the curvature radii R_1, R_2 of the coordinate lines for the introduced orthogonal system of coordinates are as follows:

$$A_1 = 1, \quad A_2 = s\,\sin\alpha, \quad R_1 = \infty, \quad R_2 = s\tan\alpha\,. \tag{2.1}$$

We restrict our consideration to the case where the directions of the orthotropy axes coincide with the directions of coordinate axes, and the structural reinforcement parameters of all the layers of the shell are independent of the angular coordinate φ but can depend on the meridional coordinate s, which is the case, for example, in the reinforcement of a conic shell along the generatrix by fibers with a constant cross section. Assuming that the shell is sufficiently thin, we ignore values of the order h/R_2 compared to unity in all equations.

The nonclassical equations of [2], which make it possible to take into account transverse shear deformations, will be used as a basis in the analysis of the free oscillations of the shell. Going from the tensor components to their physical constituents and from the covariance derivatives to partial derivatives in the tensor equations [2], performing transformation (1.1), and taking into account inequalities (2.1), we obtain a closed system of linearized differential equations for the problem on the free oscillations of a conic shell. This system includes the following groups of dependences:

The relations of elasticity (the brackets at the indices of the physical constituents are omitted, $k = 1, 2, \ldots, m$ is the ordinal number of the layer)

$$\begin{gathered} \sigma_{11}^{(k)} = a_{11}^{(k)}\varepsilon_{11}^{(k)} + a_{12}^{(k)}\varepsilon_{22}^{(k)}, \qquad \sigma_{12}^{(k)} = a_{33}^{(k)}\gamma_{12}^{(k)}\,, \\ \sigma_{22}^{(k)} = a_{12}^{(k)}\varepsilon_{11}^{(k)} + a_{22}^{(k)}\varepsilon_{22}^{(k)}\,, \qquad \tau_{13}^{(k)} = c_{11}^{(k)}\gamma_{13}^{(k)}\,, \qquad \tau_{23}^{(k)} = c_{22}^{(k)}\gamma_{23}^{(k)}\,. \end{gathered} \tag{2.2}$$

The law of distribution of the physical constituents of the displacement vector over the thickness of the package of layers

$$
\begin{gathered}
v_1^{(k)} = u_1 - z\,\frac{\partial w}{\partial s} + \mu_{11}^{(k)}\,\pi_1\,, \quad v_2^{(k)} = u_2 - \frac{z}{A_2}\,\frac{\partial w}{\partial \varphi} + \mu_{22}^{(k)}\,\pi_2\,, \quad v_3^{(k)} = w\,, \\
\mu_{ii}^{(k)}(s,z) = \frac{f(z) - f(h_{k-1})}{c_{ii}^{(k)}} + \sum_{j=1}^{k-1} \frac{f(h_j) - f(h_{j-1})}{c_{ii}^{(j)}} \quad (i = 1,2).
\end{gathered}
\tag{2.3}
$$

The deformation–displacement relations

$$
\begin{gathered}
\varepsilon_{11}^{(k)} = \frac{\partial u_1}{\partial s} - z\,\frac{\partial^2 w}{\partial s^2} + \mu_{11}^{(k)}\,\frac{\partial \pi_1}{\partial s} + \frac{\partial \mu_{11}^{(k)}}{\partial s}\,\pi_1\,, \\
\varepsilon_{22}^{(k)} = \frac{1}{A_2}\left[\frac{\partial u_2}{\partial \varphi} - \frac{z}{A_2}\,\frac{\partial^2 w}{\partial \varphi^2} + \mu_{22}^{(k)}\,\frac{\partial \pi_2}{\partial \varphi} + \sin\alpha\Big(u_1 - z\,\frac{\partial w}{\partial s} + \mu_{11}^{(k)}\,\pi_1\Big)\right] + \frac{w}{R_2}\,, \\
\gamma_{12}^{(k)} = \frac{1}{A_2}\left[\frac{\partial u_1}{\partial \varphi} - z\,\frac{\partial^2 w}{\partial s\,\partial \varphi} + \mu_{11}^{(k)}\,\frac{\partial \pi_1}{\partial \varphi} - \sin\alpha\left(u_2 - \frac{z}{A_2}\,\frac{\partial w}{\partial \varphi} + \mu_{22}^{(k)}\,\pi_2\right)\right] \\
+\frac{\partial u_2}{\partial s} - z\,\frac{\partial}{\partial s}\left(\frac{1}{A_2}\,\frac{\partial w}{\partial \varphi}\right) + \mu_{22}^{(k)}\,\frac{\partial \pi_2}{\partial s} + \frac{\partial \mu_{22}^{(k)}}{\partial s}\,\pi_2\,, \quad \gamma_{13}^{(k)} = \frac{f'(z)}{c_{11}^{(k)}}\,\pi_1\,, \quad \gamma_{23}^{(k)} = \frac{f'(z)}{c_{22}^{(k)}}\,\pi_2\,.
\end{gathered}
\tag{2.4}
$$

The dependences between the generalized internal forces and moments in the shell surface and the internal stresses in its layers

$$
\begin{gathered}
\|\, T_{\alpha\beta}, M_{\alpha\beta}, S_{\alpha\beta} \,\| = \sum_{k=1}^{m} \int_{h_{k-1}}^{h_k} \sigma_{\alpha\beta}^{(k)} \,\|\, 1, z, \mu_{\beta\beta}^{(k)} \,\|\, dz\,, \\
Q_i = \sum_{k=1}^{m} \int_{h_{k-1}}^{h_k} \left[\sigma_{1i}^{(k)}\,\frac{\partial \mu_{ii}^{(k)}}{\partial s} + \sigma_{2j}^{(k)}\,\frac{\sin\alpha}{A_2}\left(\mu_{11}^{(k)} - \mu_{22}^{(k)}\right) + \tau_{i3}^{(k)}\,\frac{f'(z)}{c_{ii}^{(k)}}\right] dz \quad (\alpha, \beta, i, j = 1,2, \quad i \neq j)\,.
\end{gathered}
\tag{2.5}
$$

The representations of the integral characteristics of d'Alembertian mass forces

$$
\|X_\beta, Y_\beta, Z_\beta, I\| = \sum_{k=1}^{m} \int_{h_{k-1}}^{h_k} \rho_k \,\|\, v_\beta^{(k)}, z v_\beta^{(k)}, \mu_{\beta\beta}^{(k)}\, v_\beta^{(k)}, w \,\|\, dz
\tag{2.6}
$$

(ρ_k is the material density of the kth layer of the shell).

The differential equations of steady-state free oscillations written in stresses and moments

$$
\begin{gathered}
\frac{\partial}{\partial s}(A_2 T_{11}) - \sin\alpha \cdot T_{22} + \frac{\partial T_{21}}{\partial \varphi} + \omega^2 A_2 X_1 = 0\,, \\
\frac{\partial T_{22}}{\partial \varphi} + \sin\alpha \cdot T_{21} + \frac{\partial}{\partial s}(A_2 T_{12}) + \omega^2 A_2 X_2 = 0\,, \\
\frac{\partial^2}{\partial s^2}(A_2 M_{11}) - \sin\alpha \cdot \frac{\partial M_{22}}{\partial s} + 2\,\frac{\partial^2 M_{12}}{\partial s \partial \varphi} + \frac{1}{A_2}\,\frac{\partial^2 M_{22}}{\partial \varphi^2} - \frac{A_2}{R_2} T_{22} \\
+ 2\,\frac{\sin\alpha}{A_2}\,\frac{\partial M_{21}}{\partial \varphi} + \omega^2\left(A_2 I + \frac{\partial}{\partial s}(A_2 Y_1) + \frac{\partial Y_2}{\partial \varphi}\right) = 0\,, \\
\frac{\partial}{\partial s}(A_2 S_{11}) - \sin\alpha \cdot S_{22} + \frac{\partial S_{21}}{\partial \varphi} - A_2 Q_1 + \omega^2 A_2 Z_1 = 0\,, \\
\frac{\partial S_{22}}{\partial \varphi} + \sin\alpha \cdot S_{21} + \frac{\partial}{\partial s}(A_2 S_{12}) - A_2 Q_2 + \omega^2 A_2 Z_2 = 0\,.
\end{gathered}
\tag{2.7}
$$

The function $f(z)$ determining the distribution law of transverse shear stresses throughout the shell thickness is taken in the form

$$f(z) = z^3 - 1.5\,h\,z^2\,, \tag{2.8}$$

corresponding to the quadratic dependence of these values on the normal coordinate z (see [2]).

Equations (2.1)–(2.8) are a complete system of nonclassical differential equations for the problem of the natural oscillations of the conic shell. The order of this system is equal to 12, which requires the setting of six boundary conditions at the domain boundary in a properly posed boundary-value problem. The complete set of variants of such conditions corresponding to different methods of fixing and loading the edges is given in [2]. In the case considered below of a shell closed in the circular direction with rigidly fixed edges $s = a$, $s = b$, these conditions require vanishing of the generalized displacements at points of the boundary contour (see [2]):

$$\text{at} \quad s = a,\ b \quad w = \frac{\partial w}{\partial s} = u_1 = u_2 = \pi_1 = \pi_2 = 0\,, \tag{2.9}$$

and also the 2π-periodicity of the solution with respect to the angular coordinate φ.

Equations (2.1)–(2.8) allow for the orthotropy of the deformability properties, the ultimate shear rigidity of all or part of the layers of the shell, and the variability of the elasticity coefficients and, therefore, can be used for the analysis of the natural oscillations of a wide class of thin-walled layered composite conic shells. The nondependence of the order and structure of these equations on the number of the shell layers and on the layer package structure as a whole should also be considered as an advantage. This simplifies the statement and investigation of the problem on the free oscillations of the multilayered shell as a linear boundary-value problem on eigenvalues for a system of partial differential equations.

Also, note the passage to the limit (see [2])

$$c_{ii}^{(k)} \to \infty \tag{2.10}$$

($k = 1, 2, \ldots, m$, $i = 1, 2$) from Eqs. (2.1)–(2.8) to the classical equations of the steady-state free oscillations of the conic shell. This passage to the limit is used below to estimate the influence of transverse shear deformations on the natural frequencies.

The dimensionless independent variable x and the vector $\mathrm{y} = [y_1, y_2, \ldots, y_{12}]^t$ of the dimensionless kinematic and force characteristics of the strain-stressed state of the shell are introduced by means of the equalities

$$\begin{gathered} x = s/b, \quad w = hy_1, \quad y_2 = y_1', \quad u_1 = by_3, \quad u_2 = by_4, \quad \pi_1 = E_1^c\, b\, y_5/h^3, \quad \pi_2 = E_1^c\, b\, y_6/h^3, \\ \frac{\partial}{\partial s}(A_2 M_{11}) - \sin\alpha \cdot M_{22} + 2\,\frac{\partial M_{12}}{\partial \varphi} + \omega^2 A_2 Y_1 = h^2 E_1^c\, y_7, \\ A_2 M_{11} = h^2 b E_1^c y_8, \quad A_2 T_{11} = hbE_1^c y_9, \quad A_2 T_{21} = hbE_1^c y_{10}\,, \quad A_2 S_{11} = h^4 b y_{11}, \quad A_2 S_{12} = h^4 b y_{12}, \end{gathered} \tag{2.11}$$

where E_1^c is the Young's modulus of the first binding (internal) layer of the shell. The equations of steady-state free oscillations (2.1)–(2.8) in the variables (2.11) and the boundary conditions of rigid fixing (2.9) can be written in the matrix form

$$\mathcal{A}\,(x, D_\varphi)\,\frac{\partial \mathrm{y}}{\partial x} = \mathcal{B}\,(x, D_\varphi)\,\mathrm{y} + \omega^2\,\mathcal{C}\,(x, D_\varphi)\,\mathrm{y}\,; \tag{2.12}$$

$$\|E_6, O_6\|\, y\,(a/b, \varphi) = 0\,, \qquad \|E_6, O_6\|\, y\,(1, \varphi) = 0\,. \tag{2.13}$$

In (2.12), $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are matrices 12×12 whose elements are polynomials of the differential operator D_φ ($D_\varphi = \partial/\partial\varphi$) with coefficients depending on the variable x. The expressions for elements of the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are not given here since they are cumbersome. We indicate only zero and nonzero columns of the matrix $\mathcal{C}$ combining the numbers of zero columns (see Section 1) into the set K, and the numbers of nonzero columns into the set J:

$$J = \{1, 2, 3, 4, 5, 6\}, \quad K = \{1, 2, \ldots\, 12\} \setminus J. \tag{2.14}$$

TABLE 1

L	$\omega_1^{(0)}$	$\omega_2^{(0)}$	$\omega_3^{(0)}$	$\omega_4^{(0)}$	$\omega_5^{(0)}$	$\omega_6^{(0)}$
	kHz					
6	1.202	2.327	2.781	3.510	3.708	4.392
8	1.202	2.327	2.781	3.465	3.693	4.114
10	1.202	2.327	2.781	3.465	3.669	4.055
12	1.202	2.327	2.781	3.465	3.668	4.034

TABLE 2

n	Classical theory			Nonclassical theory		
	$\omega_1^{(n)}$	$\omega_2^{(n)}$	$\omega_3^{(n)}$	$\omega_1^{(n)}$	$\omega_2^{(n)}$	$\omega_3^{(n)}$
	kHz					
0	1.202	2.328	2.782	1.202	2.327	2.781
2	0.515	0.954	1.486	0.513	0.949	1.475
4	0.634	0.954	1.318	0.628	0.941	1.294
6	1.079	1.471	2.128	1.057	1.431	2.047
8	1.717	2.293	3.481	1.661	2.196	3.270
10	2.525	3.386	5.174	2.408	3.186	4.724

Note that after deletion of the 5th. 6th. 11th, and 12th rows and of the same columns from the 12×12 matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, the corresponding 8×8 matrices of coefficients of the classical system of differential equations for the free oscillations of the conic shell are obtained. This follows immediately from the passage to the limit (2.9), if we take into account that these rows and columns of matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ vanish as $c_{ii}^{(k)} \to \infty$.

The solution to problem (2.12). (2.13) is constructed in the form of the Fourier series

$$y = u_0(x) + \sum_{n=1}^{\infty} (u_n(x) \cos n\varphi + v_n(x) \sin n\varphi) \tag{2.15}$$

in terms of a system of trigonometric functions with the vector coefficients $u_n(x)$, $v_n(x)$. It is clear that the solution in the form (2.15) satisfies the condition of 2π-periodicity with respect to the angular coordinate φ, and also the boundary conditions of rigid fixing (2.13) if the vector-functions $u_n(x)$, $v_n(x)$ satisfy the requirement

$$\|E_6, O_6\|\, u_n = \|E_6, O_6\|\, v_n = 0 \quad \text{at} \quad x = a/b, 1. \tag{2.16}$$

Substituting this solution into (2.12), setting the general term of the Fourier series obtained equal to zero, and taking into account the boundary conditions (2.16), we obtain two linear boundary-value problems of the form (1.3) for the coefficients of expansion (2.15), which are not related to each other. It can be shown that the eigenvalues of these boundary-value problems coincide, and the eigenvectors corresponding to them are obtained from one another by a linear orthogonal transformation of revolution. Therefore, it will suffice to consider only one of these problems.

Thus, the investigation of the free oscillations of the conic shell is reduced to the integration of a linear boundary-value problem on the eigenvalues of a system of ordinary differential equations. The numerical solution to this problem was obtained by the method developed in Section 1 by using the orthonormal coordinate system

$$\psi_{kj}(x) = \sqrt{\frac{2k-1}{1-a/b}}\, P_{k-1}\left(2\frac{x-a/b}{1-a/b} - 1\right) \mathbf{e}_j\,, \qquad k = 1, 2, \ldots, L, \;\; j \in J\,, \tag{2.17}$$

where $P_k(t)$ are the Legendre polynomials orthogonal in the segment $[-1, 1]$ and $\mathbf{e}_j$ are the vectors of the standard orthonormal basis in $\mathbf{R}^{12}$. It is seen from (2.14) that the coordinate system (2.17) consists of $6L$

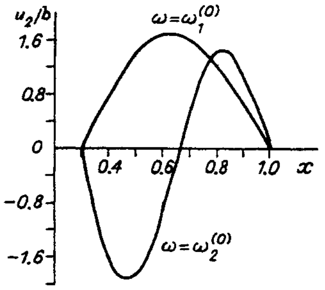

Fig. 1

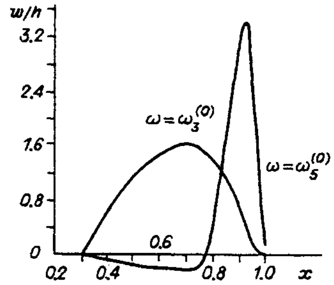

Fig. 2

vectors, which requires the solution of boundary-value problem (1.11) for the $12 \times 6L$ matrix $Z(x)$ and the solution of the algebraic eigenvalue problem for the $6L \times 6L$ matrix of coefficients of system (1.8). Boundary-value problem (1.11) was solved by the method of invariant imbedding [1], and the QR-algorithm in combination with a preliminary reduction of the matrix to the Hessenberg form was used in the numerical solution of the eigenvalue problem (see [10]). The value of the parameter L sufficient to provide high accuracy of the result was determined by means of numerical investigation of the convergence rate of the method. The corresponding numerical data are cited below.

The calculations were carried out on an Elbrus-2 computer.

3. Numerical Results. Table 1 presents the data which make it possible to evaluate the convergence rate of the method relative to the parameter L, and the corresponding values of six lower natural frequencies $\omega_1^{(0)}, \omega_2^{(0)}, \ldots, \omega_6^{(0)}$ of axisymmetric forms of oscillations found on the basis of nonclassical equations (2.1)–(2.8). The results were obtained for a two-layered composite shell whose internal layer was reinforced by fibers of constant cross section in the circular direction, and the external layer was reinforced in the meridional direction at the geometrical parameters

$$b = 1\text{-m}, \quad \alpha = \pi/10, \quad a/b = 0.3, \quad h/b = 0.01, \quad h_1 - h_0 = h_2 - h_1 = 0.5h; \tag{3.1}$$

mechanical parameters

$$\begin{gathered} E_1^{\mathrm{b}} = E_2^{\mathrm{b}} = 3000 \text{ MPa}, \quad E_1^{\mathrm{r}} = E_2^{\mathrm{r}} = 250 \text{ GPa}, \quad \nu_1^{\mathrm{b}} = \nu_2^{\mathrm{b}} = \nu_1^{\mathrm{r}} = \nu_2^{\mathrm{r}} = 0.3, \\ \rho_1^{\mathrm{b}} = \rho_2^{\mathrm{b}} = 1250 \text{ kg/m}^3, \quad \rho_1^{\mathrm{r}} = \rho_2^{\mathrm{r}} = 1710 \text{ kg/m}^3, \end{gathered} \tag{3.2}$$

and structural ones

$$\omega_{z1} = \omega_{z2} = 0.5, \quad \omega_1 = 0.5, \quad \omega_2\,|_{x=a/b} = 0.9 \tag{3.3}$$

of the composite layered shell. Here $h_k - h_{k-1}$, ω_k, ω_{zk} $(k = 1, 2)$ are, respectively, the thickness of the kth layer and the reinforcement intensity in its surface and over its thickness (see [15]): $E_k^{\mathrm{b(r)}}$, $\nu_k^{\mathrm{b(r)}}$, $\rho_k^{\mathrm{b(r)}}$ is the Young's modulus, the Poisson's coefficient, and the density of the binding material (index b) and reinforcing (index r) fibers of the kth layer. The values of the mechanical parameters (2.3) correspond (see [16]) to epoxy binding fibers and boron reinforcing fibers. The effective rigidity and pliability of the layers were determined on the basis of equations for the model of a reinforced layer developed in [15].

It is seen from Table 1 that the approximations to the exact values of natural frequencies in the process being considered are realized from above. Stabilization of the computation process for three lower natural frequencies is achieved even at $L = 6$, for the 4th natural frequency, at $L = 8$, for the 5th, at $L = 10$, and for the 6th, at $L = 12$. Similar regularities are also valid for nonaxisymmetric forms of oscillation, as was revealed by numerical investigation. Note that in the example considered, the axisymmetric forms of free oscillations corresponding to the natural frequencies $\omega_1^{(0)}, \omega_2^{(0)}, \omega_4^{(0)}$ are twisting forms, and the forms corresponding to the frequencies $\omega_3^{(0)}, \omega_5^{(0)}, \omega_6^{(0)}$ are predominantly bending forms.

The twisting forms of oscillations corresponding to the frequencies $\omega_1^{(0)}$, $\omega_2^{(0)}$ are shown in Fig. 1 by circular displacements of the reference surface, and predominantly bending forms corresponding to the frequencies $\omega_3^{(0)}$, $\omega_5^{(0)}$ are shown in Fig. 2 by deflections,

Table 2 lists. depending on the parameter of circular wave formation n, the calculation results for three lower natural frequencies found both on the basis of nonclassical equations (2.1)–(2.8) and classical equations (2.1)–(2.8), and (2.10) for the steady-state free oscillations of the conic shell. The results were obtained with values of the parameters in (3.1)–(3.3).

It is seen from Table 2 that neglect of transverse shear deformations leads to overestimation of the calculated values of natural frequencies; the overestimation increases with increase in the ordinal number of the frequency being determined and with increase in the number n of circular harmonics being considered. Specifically, the relative error introduced into the determination of the natural frequency $\omega_1^{(0)}$ by neglecting transverse shears is practically absent, whereas this error comprises even 4.63% in the determination of the natural frequency $\omega_1^{(10)}$. In the determination of the natural frequencies $\omega_3^{(0)}$ and $\omega_3^{(10)}$, the relative error owing to the neglect of transverse shear deformations is, respectively, 0.04 and 8.70%. It is also seen from Table 2 that the dependences of the lower natural frequencies of the conic shell on the number of circular harmonics n are characterized by the presence of minimum points. A similar result was obtained in [7], where the corresponding analysis was carried out for a composite cylindrical shell.

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